Off-by-Ones / Fencepost Errors

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PROBLEM 1. I'm in math class and have been assigned problems 3–15 for tomorrow. How many problems do I have to do?

If you subtracted 3 from 15 and got 12, you're wrong — but you're in good company, because this is one of the easiest errors to make in arithmetic.

If you're not sure why, let's count them:

#	3	4	5	6	7	8	9	10	11	12	13	14	15
Count	1	2	3	4	5	6	7	8	9	10	11	12	13

PROBLEM 1.5. Here's another way of looking at it: let's change the problem. I have to do problems 1–10. How many are there?

Here the answer is 10. Even if you got the last one wrong, you almost certainly got this one right. Why? Because 1 is at the end, you know intuitively that the other end is the number of items — it's like you're counting. You wouldn't subtract 1 from 10 and get 9. So why in the other one?

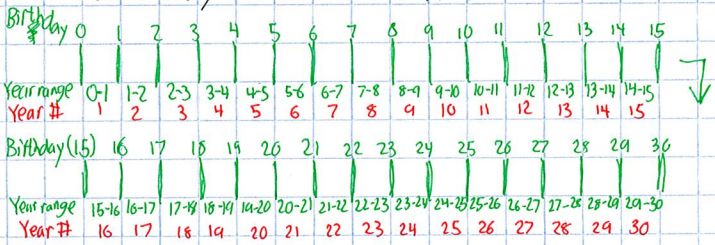
PROBLEM 2. Here's another classic. You want to build a 40-foot-long fence with posts every 5 feet. How many posts do you need?

We can draw this one: (□ = 5 ft)

So $40/5 = 8$, but we need 9 posts, because we counted distance, not endpoints. (This particular problem gives this type of "fencepost" error its name.)

PROBLEM 3. One more. You have your 30th birthday today. How many years have you lived?

This one is interesting because people sometimes erroneously include the endpoints:



So your 30th birthday occurs on the day you've been alive for 30 years, which makes sense. You're then *in* your 31st year. The reason this is confusing is because the day you were born is your zeroth birthday, but we start counting from 1, so during your first year of life, you are 0 years old.

How to Fix It (and Think Clearly)

A foolproof way to get it right, of course, is to count. But that's hardly practical when you have a range like 17–2,463, and it's annoying even when it is practical. This is what arithmetic was invented for, after all! Here's a better way.

Exclusive Counting
Range: 1–5 ($1 < x < 5$)
1 | 2 3 4 | 5
Count: 3
By subtraction: 4

Inclusive Counting
Range: 3–8 ($3 \le x \le 8$)
| 3 4 5 6 7 8 |
Count: 6
By subtraction: 5

Inclusive on One End
Range: 2–7 ($2 < x \le 7$)
2 | 3 4 5 6 7 |
Count: 5
By subtraction: 5

From here I think it should be clear that subtraction is inclusive on one end and exclusive on the other.

* **So the Simple Rule: Rule of Ranges**
When counting the number of elements in a range, subtract one less than the lower bound (e.g. 15 − (3−1) or 15 − 2 = 13 for the initial problem).

Why?

Have a look at this diagram of the range 4–9: (inclusive)

(1) (2) (3) 4 5 6 7 8 9] 9 elements

To get the right answer for, say, 9−5, we have to remove 5 elements, including the 5 if we take from the bottom. The 9 is untouched, though.

If this (one) makes sense to you, great. If not, no worries:

To put it another way, adding 5 + 4:

1 2 3 4 5 | 1 2 3 4
1 2 3 4 5 6 7 8 9

If we wanted subtraction to be inclusive, we'd have to add another number here, or work from "above 9" to subtract 5, which is obviously wrong.

Off-by-Ones / Fencepost Errors 2: Avoiding Strategies

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Answer to #1) At the end of page 1, I presented the Rule of Ranges. It works because subtraction is inclusive on one end and exclusive on the other. When we count the number of items in a list, we want both ends inclusive:

Number	1	2	3	4	5	6	7	8	COUNT
Inclusive	1	2	3	4	5	6	7	8	8
One of Each	1	2	3	4	5	6	7	–	7
Exclusive	–	1	2	3	4	5	6	–	6

All that's needed to move "one of each" to "inclusive" is to add one. So you could accomplish Problem 1 this way:

(Upper Bound – Lower Bound) + 1
15 – 3 + 1 = 13

... which is of course mathematically equivalent to the Rule of Ranges:

Upper Bound – (Lower Bound – 1)

I prefer the latter formulation – it just seems more elegant – but the former will always get you the same answer, so if you like it better, go for it.

ANSWER TO #2

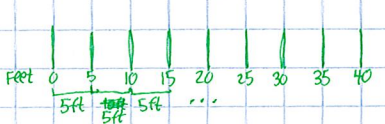
This one we can't handle with the Rule of Ranges, as we're not placing fenceposts each foot. We can convert units first, though:

1 fence-distance / 5 feet · 40 feet to cover = 8 fence-distances

This is technically a range of 8–0 fence-distances, so we can actually get the problem right with the Rule of Ranges:

8 – (0 – 1) = 8 – –1 = 8 + 1 = 9 fenceposts

But this is silly. It's far better to adapt our intuition to this difficult-to-intuit field of off-by-ones. When we say the distance is 40 feet, recognize we're by definition excluding an endpoint.



Each distance is a subtraction problem with one end excluded, as is the entire thing (which is non-obvious when it starts at 0, but start at 5 and go to 45 and it's clearer). So we simply need to realize that to make it inclusive on both ends, we add 1.

Calculate # of posts: 40/5 = 8 (exclusive on one end)
Make doubly inclusive: 8 + 1

→ To make a long story short: Add 1 if you've subtracted in any way and have not directly used the Rule of Ranges.

Answer to #3

This one is a little different because it's based on zero-based counting. So on your 30th birthday you've both lived 30 years and had 30 birthdays past 0. (If we were 1 on the day we were born, we'd only have lived 29 years, but had 30 birthdays past 0 – the situation in most cases, like problem #2.)

An easy rule for birthdays and anniversaries:
- After your n^{th} birthday, you are in your $n+1$ year of life.
- The day you turn n is also your n^{th} birthday.